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Expansion of Jacobi polynomials in terms of orthogonal functions

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Abstract. Polynomials $P_n^{\alpha,\beta}(\psi)$, where ψ is the angle between the unit vectors $\hat{r}_1=(1,\theta_1,\phi_1)$ and $\hat{r}_2=(1,\theta_2,\phi_2)$, are given by two representations. The calculation of the coefficients in the suggested representations explicitly justifies the forms given by equations (6) and (12). The link between the two representations provides two interesting summation rules.

1. Introduction

Jacobi polynomials $P_n^{\alpha,\beta}(x)$ form a larger class of orthogonal polynomials than Gegenbauer polynomials $C_n^{\alpha+1/2}(x)$, Chebyshev polynomials $T_n(x)$ and Legendre polynomials $P_n(x)$. They are related by (see equations (8.962.2) and (8.962.3) in [1]):

$$P_n^{\alpha,\alpha}(x)=\frac{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)\Gamma(\alpha+1)}\,C_n^{\alpha+1/2}(x) \tag{1}$$

$$P_n^{-1/2,-1/2}(x)=\frac{(2n)!}{2^{2n}(n!)^2}\,T_n(x) \tag{2}$$

$$P_n^{0,0}(x)=P_n(x). \tag{3}$$

On using equation (8.932.1) in [1] we write equation (1) as

$$P_m^{\alpha,\alpha}(x)=\frac{\Gamma(2\alpha+1)\Gamma(m+\alpha+1)}{\Gamma(m+2\alpha+1)\Gamma(\alpha+1)}\sum_z\frac{(-1)^z\Gamma(m+\alpha+\frac{1}{2}-z)(2x)^{m-2z}}{\Gamma(\alpha+\frac{1}{2})z!(m-2z)!}. \tag{4}$$

For $\alpha>-1$, $\beta>-1$ the polynomials $P_n^{\alpha,\beta}(x)$ satisfy the orthogonality relation (see equation (7.391.1) in [1])

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)\,\mathrm{d}x=\delta_{nm}\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(\alpha+\beta+2n+1)\Gamma(n+\alpha+\beta+1)}. \tag{5}$$

Koornwinder [1, 2] studied the special case when α, β are integers or half-integers and obtained an addition formula for these special polynomials using group theoretical methods [4].

In this paper we tackle the general case where α and β can take any value greater than -1 and propose the addition formula

$$\begin{aligned}P_{2n+k}^{\alpha,\beta}&(\cos\psi)\\&=\sum_{L,m=0}^{2n+k}A(L,m)(\sin\theta_1\sin\theta_2)^m(\cos\theta_1\cos\theta_2)^{2n+2k-2m}P_{2n+k-L}^{\alpha_1,L+\beta_1}(\cos\theta_1)\\&\quad\times P_{2n+k-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2)P_m^{\alpha_3,\beta_3}(\cos\phi)\end{aligned} \tag{6}$$

which suits both even and odd degrees ($k=0$ or 1), and find that

$$A(L,m)=\frac{\sqrt{\pi}\,(-1)^{k-L}\Gamma(2n+k+L+\alpha_2+\beta_2+1)\Gamma(2n+k+\alpha+1)(4n+2k+\alpha_2+\beta_2+1)}{2^{L+\beta_2}\Gamma(2n+k+\alpha+\beta+1)\Gamma(2n+k+\beta_2+1)C(m)K(2n+k-L)}$$
$$\times\sum_{rjy}\frac{(-1)^{r+j+y-m}\Gamma(2n+k+\alpha+\beta+r+1)\Gamma(2n+k+\beta_2+y+1)j!\,\Gamma(j+\frac{1}{2})}{2^{r+y-2j}(2n+k-r)!\Gamma(\alpha+r+1)(r+m-2n-2k-2j)!(2j+2n+2k-2m)!\Gamma(j+\frac{3}{2})}$$
$$\times\frac{1}{\Gamma(\frac{1}{2}-j)y!\Gamma(2n+k+\alpha_2-y+1)\Gamma(2j+L+\beta_2+y+2)} \tag{7}$$

where

$$\cos\psi=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi \tag{8}$$

$$\phi=\phi_1-\phi_2 \tag{9}$$

$$C(m)=\frac{\Gamma(2\alpha_3+1)\Gamma(m+\alpha_3+1)\Gamma(m+\alpha_3+\frac{1}{2})2^m}{\Gamma(m+2\alpha_3+1)\Gamma(\alpha_3+1)\Gamma(\alpha_3+\frac{1}{2})} \tag{10}$$

and

$$K(2n+k-L)=\frac{2(-1)^{k-L}\Gamma(2n+k+\beta_1+1)}{(2n+k-L+1)!\Gamma(L+\beta_1)(2n+k+\alpha_1+\beta_1+1)}. \tag{11}$$

We will also express Jacobi polynomials in terms of the spherical harmonics $Y_p^m(\theta_1,\phi_1)$ and $Y_p^m(\theta_2,\phi_2)$ in the form

$$P_n^{\alpha,\beta}(\cos\psi)=\sum_{pm}B(p,n)Y_p^{*m}(\theta_1,\phi_1)Y_p^m(\theta_2,\phi_2) \tag{12}$$

where

$$B(p,n)=\frac{4\pi\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}\sum_r\frac{(-1)^{r+p}r!\Gamma(n+r+\alpha+\beta+1)}{(n-r)!(r-p)!(r+p+1)!\Gamma(\alpha+r+1)}. \tag{13}$$

Finally, we compare the coefficients of $e^{im\phi}$ when $m=2n+k$ and $m=0$ in equations (6) and (12) to get the two summation rules

$$\sum_{Ly}\frac{(-1)^y(2n+k-L+1)!\Gamma(2n+k+L+\alpha_2+\beta_2+1)\Gamma(2n+k+\beta_2+y+1)\Gamma(L+\beta_1)}{2^{y+L}y!\Gamma(2n+k+\alpha_2-y+1)\Gamma(2n+k+L+\beta_2+y+2)}$$
$$\times P_{2n+k-L}^{\alpha_1,L+\beta_2}(\cos\theta_1)P_{2n+k-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2)$$
$$=\frac{2^{\beta_2}(2n+1)\Gamma(2n+k+\beta_1+1)\Gamma(2n+k+\beta_2+1)}{(4n+2k+\alpha+\beta+1)(2n+\alpha_1+\beta_1)(2n)!(2n+2K)!}(\cos\theta_1\cos\theta_2)^{2n} \tag{14}$$

where $k=0$ or 1, and

$$\sum_{pr}\frac{(-1)^{p+r}r!(2p+1)\Gamma(n+\alpha+\beta+r+1)}{(n-r)!(r-p)!(r+p+1)!\Gamma(r+\alpha+1)}P_p(\cos\theta_2)P_p(\cos\theta_2)$$
$$=\frac{(-1)^n\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+1)}(\cos\theta_1\cos\theta_2)^n. \tag{15}$$

2. Calculation of the coefficients $A(L,m)$

We use equation (8.962.1) in [1] as

$$\hat{P}_n^{\alpha,\beta}(x)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{rq}\frac{(-1)^{r+q}\Gamma(n+r+\alpha+\beta+1)_x^q}{2^r(n-r)!\Gamma(r+\alpha+1)q!(r-q)!} \tag{16}$$

and equations (4) and (8) to write equation (6) as

$$\frac{\Gamma(2n+k+\alpha+1)}{\Gamma(2n+k+\alpha+\beta+1)}$$

$$\times\sum_{rqv}\frac{(-1)^{r+q}\Gamma(2n+k+r+\alpha+\beta+1)(\cos\theta_1\cos\theta_2)^{q-v}(\sin\theta_1\sin\theta_2)}{2^r(2n+k-r)!(r-q)!v!(q-v)!\Gamma(r+\alpha+1)}(\cos\phi)^v$$

$$=\sum_{Lmz}A(L,m)(\sin\theta_1\sin\theta_2)^m(\cos\theta_1\cos\theta_2)^{2n+2k-2m}P_{2n+k-L}^{\alpha_1,L+\beta_1}(\cos\theta_1)$$

$$\times P_{2n+k-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2)$$

$$\times\frac{\Gamma(2\alpha_3+1)\Gamma(m+\alpha_3+1)(-1)^z\Gamma(m+\alpha_3+\frac{1}{2}-z)(2\cos\phi)^{m-2z}}{\Gamma(m+2\alpha_3+1)\Gamma(\alpha_3+1)\Gamma(\alpha_3+\frac{1}{2})z!(m-z)!}.$$

Comparing coefficients of $\cos\phi^m$ on both sides we get

$$\sum_L A(L,m)C(m)P_{2n+k-L}^{\alpha_1,L+\beta_1}(\cos\theta_1)P_{2n+k-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2)$$

$$=\frac{\Gamma(2n+k+\alpha+1)}{\Gamma(2n+k+\alpha+\beta+1)}$$

$$\times\sum_{rq}\frac{(-1)^{r+q}\Gamma(2n+k+r+\alpha+\beta+1)(\cos\theta_1\cos\theta_2)^{q+m-2n-2k}}{2^r(2n+k-r)!(r-q)!(q-m)!\Gamma(\alpha+r+1)}. \qquad (17)$$

Integrating both sides of equation (17) with respect to θ_1, the resulting integral on the left-hand side is evaluated using equation (7.391.2) in [1] in the form

$$\int_0^\pi P_{2n+k-L}^{\alpha_1,L+\beta_1}(\cos\theta_1)\sin\theta_1\,d\theta_1$$

$$=\frac{2(-1)^{k-L}\Gamma(2n+k+\beta_1+1)}{(2n+k+1-L)!\Gamma(L+\beta_1)(2n+k+\alpha_1+\beta_2)}$$

$$=K(2n+k-L) \qquad (18)$$

where

$$K(0)=2. \qquad (19)$$

We also notice that the resulting integral on the right-hand side of equation (17) is of the form

$$\int_0^\pi(\cos\theta_1)^{q+m-2n-2k}\sin\theta_1\,d\theta_1$$

which vanishes unless $(q+m-2n-2k)$ is an even integer $2j$. Under these conditions equation (17) is written as

$$\sum_L A(L,m)C(m)K(2n+k-L)P_{2n+k-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2)$$

$$=\frac{\Gamma(2n+k+\alpha+1)}{\Gamma(2n+k+\alpha+\beta+1)}$$

$$\times\sum_{rj}\frac{(-1)^{r-m}\Gamma(2n+k+r+\alpha+\beta+1)\Gamma(j+\frac{1}{2})(\cos\theta_2)^{2j}}{2^r(2n+k-r)!\Gamma(\alpha+r+1)(r+m-2n-2k-2j)!(2j+2n+2k-2m)!\Gamma(j+\frac{3}{2})}. \qquad (20)$$

Finally, we use equation (5) to invert equation (20) to the form

$$
\begin{aligned}
A(L, m) = {} & \frac{(2n+k-L)!(4n+2k+\alpha_2+\beta_2+1)\Gamma(2n+k+L+\alpha_2+\beta_2+1)\Gamma(2n+k+\alpha+1)}{2^{2L+\alpha_2+\beta_2+1}\Gamma(2n+k+\alpha_2+1)\Gamma(2n+k+\beta_2+1)\Gamma(2n+k+\alpha+\beta+1)C(m)} \\
& \times \frac{1}{K(2n+k-L)} \\
& \times \sum_{rjs} \frac{(-1)^{r+m+s}\Gamma(2n+k+r+\alpha+\beta+1)j!\Gamma(j+\frac{1}{2})}{2^r(2n+k-r)!\Gamma(r+\alpha+1)(r+m-2n-2k-2j)!(2j+2n+2k-2m)!} \\
& \times \frac{1}{\Gamma(j+\frac{3}{2})s!(j-s)!} \int_{-1}^{1} (1-x)^{L+\alpha_2+s}(1+x)^{L+\beta_2+s} P_{2n+k-L}^{L+\alpha_2, L+\beta_2}(x)\, \mathrm{d}x \\
= {} & \frac{(2n+k-L)!(4n+2k+\alpha_2+\beta_2+1)\Gamma(2n+k+\alpha+1)}{\Gamma(2n+k+\beta_2+1)\Gamma(2n+k+\alpha+\beta+1)C(m)K(2n+k-L)} \\
& \times \sum_{rj} \frac{(-1)^{r-m}\Gamma(2n+k+r+\alpha+\beta+1)j!\Gamma(j+\frac{1}{2})}{2^r(2n+k-r)!\Gamma(r+\alpha+1)(r+m-2n-2k-2j)!(2j+2n+2k-2m)!\Gamma(j+\frac{3}{2})} \\
& \times \sum_{xs} \frac{(-1)^{s+x}2^{2s}\Gamma(2n+k+L+\alpha_2+\beta_2+x+1)\Gamma(L+\alpha_2+s+x+1)\Gamma(L+\beta_2+s+1)}{x!s!(j-s)!(2n+k-L-x)!\Gamma(L+\alpha_2+x+1)\Gamma(2L+\alpha_2+\beta_2+2s+x+2)}
\end{aligned}
\tag{21}
$$

where we have used equation (7.391.2) in [1] again. Substituting equation (A2) (see the appendix) for the summation over s and x in equation (21) we immediately arrive at the required result of equation (7).

3. Calculation of the coefficients $B(p, n)$

Using the orthonormality of the spherical harmonics we invert equation (12) to the form

$$
\begin{aligned}
B(p, n) = {} & \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} P_n^{\alpha,\beta}(\cos\psi) Y_p^m(\theta_1, \phi_1) Y_p^{*m}(\theta_2, \phi_2) \\
& \times \sin\theta_1 \sin\theta_2 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\phi_1 \, \mathrm{d}\phi_2 .
\end{aligned}
\tag{22}
$$

Since $B(p, n)$ is independent of m we choose $m = p$ and use equation (2.2.5) in [5] as

$$
Y_p^p(\theta, \phi) = \frac{(-1)^p}{2^p p!} \left(\frac{(2p+1)!}{4\pi} \right)^{1/2} (\sin\theta \, \mathrm{e}^{\mathrm{i}\phi})^p \tag{23}
$$

and equation (16) to write equation (22) in the form

$$
\begin{aligned}
B(p, n) = {} & \frac{(2p+1)!\Gamma(n+\alpha+1)}{4\pi 2^p \Gamma(n+\alpha+\beta+1)p!p!} \\
& \times \sum_{rqht} \frac{(-1)^{r+q}\Gamma(n+r+\alpha+\beta+1)}{2^{r+h}(n-r)!\Gamma(r+\alpha+1)(r-q)!(q-h)!(h-t)!} \\
& \times \frac{1}{t!} \int_0^{\pi} (\cos\theta_1)^{q-h}(\sin\theta_1)^{h+p+1} \, \mathrm{d}\theta_1 \int_0^{\pi} (\cos\theta_2)^{q-h}(\sin\theta_2)^{h+p+1} \, \mathrm{d}\theta_2 \\
& \times \int_0^{2\pi} \int_0^{2\pi} \exp[\mathrm{i}(h-2t+p)(\phi_1-\phi_2)] \, \mathrm{d}\phi_1 \, \mathrm{d}\phi_2
\end{aligned}
\tag{24}
$$

which shows that the integrals vanish unless $t=(p+h)/2$ and $q-h$ is an even integer. Under these conditions the value of the integrals after carrying out the summation over h is

$$\frac{4\pi^3 p!\,q!}{2^q\Gamma(p+\frac{3}{2})[(q-p)/2]!\,\Gamma[\frac{1}{2}(p+q+3)]}.$$

Equation (24) now takes the form

$$B(p,n)=\frac{2\pi^{3/2}\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)} \times\sum_{rq}\frac{(-1)^{q+r}\Gamma(n+r+\alpha+\beta+1)}{2^{r+q}(n-r)!(r-q)![(q-p)/2]!\,\Gamma(r+\alpha+1)\Gamma[(p+q+3)/2]} \tag{25}$$

where we have used the duplication formula for the gamma function as

$$\Gamma(2x)=\frac{2^{2x-1}}{\pi^{1/2}}\Gamma(x)\Gamma(x+\tfrac{1}{2}). \tag{26}$$

The expression given by equations (12) and (13) is expected to be more convenient for handling expansions about a displaced centre. Such an expression would enable us to evaluate physical quantities like electronic energy, electric dipole transition probabilities, molecular dipole moments and various other quantities which are usually expressed in terms of integrals of the transition operator known as the multicentre integrals.

4. The link between the two representations

We seek first to equate the coefficients of $e^{i(2n+k)\phi}$ in equations (6) and (12). From equation (6) this coefficient equals F, and is

$$\begin{aligned}F=&\sum A(L,2n+k)\frac{C(2n+k)}{2^{2n+k}}(\sin\theta_1\sin\theta_2)^{2n+k}(\cos\theta_1\cos\theta_2)^{-2n}\\ &\times P^{\alpha_1,L+\beta_1}_{2n+k-L}(\cos\theta_1)P^{L+\alpha_2,L+\beta_2}_{2n+k-L}(\cos\theta_2)\\ =&\frac{(\sin\theta_1\sin\theta_2)^{2n+k}(\cos\theta_1\cos\theta_2)^{-2n}(\pi)^{1/2}\Gamma(2n+k+\alpha+1)(4n+2k+\alpha_2+\beta_2+1)}{2^{2n+k}\Gamma(2n+k+\alpha+\beta+1)\Gamma(2n+k+\beta_2+1)}\\ &\times\sum_L\frac{(-1)^L\Gamma(2n+k+L+\alpha_2+\beta_2+1)}{2^{L+\beta_2}K(2n+k-L)}P^{\alpha_1,L+\beta_1}_{2n+k-L}(\cos\theta_1)P^{L+\alpha_2,L+\beta_2}_{2n+k-L}(\cos\theta_2)\\ &\times\sum_{rjy}\frac{(-1)^{r+j+y}\Gamma(2n+k+\alpha+\beta+r+1)\Gamma(2n+k+\beta_2+y+1)j!\,\Gamma(j+\frac{1}{2})}{2^{r+y-2j}(2n+k-r)!\,\Gamma(r+\alpha+1)(r-k-2j)!\,(2j-2n)!\,\Gamma(j+\frac{3}{2})\Gamma(\frac{1}{2}-j)}\\ &\times\frac{1}{y!\,\Gamma(2n+k+\alpha_2-y+1)\Gamma(2j+L+\beta_2+y+1)}\end{aligned} \tag{27}$$

where we have used equation (7).

The factorials $(r-k-2j)!$ and $(2j-2n)!$ appearing in equation (27) dictate the values $r=2n+k$ and $j=n$. Therefore, equation (27) is written as

$$\begin{aligned}F=&\frac{(4n+2k+\alpha_2+\beta_2+1)(2n+k+\alpha_1+\beta_1)(2n)!\,\Gamma(4n+2k+\alpha+\beta+1)(\sin\theta_1\sin\theta_2)^{2n+k}}{2^{4n+2k+\beta_2}(2n+1)\Gamma(2n+k+\alpha+\beta+1)\Gamma(2n+k+\beta_2+1)(\Gamma(2n+k+\beta+1)}\\ &\times(\cos\theta_1\cos\theta_2)^{-2n}\\ &\times\sum_{Ly}\frac{(-1)^y(2n+k-L+1)!\,\Gamma(L+\beta_1)\Gamma(2n+k+L+\alpha_2+\beta_2+1)}{2^{L+y}y!\,\Gamma(2n+k+\alpha_2-y+1)\Gamma(2n+k+L+\beta_2+1)}\\ &\times\Gamma(2n+k+\beta_2+y+1)P^{\alpha_1,L+\beta_1}_{2n+k-L}(\cos\theta_1)P^{L+\alpha_2,L+\beta_2}_{2n+k-L}(\cos\theta_2)\end{aligned} \tag{28}$$

where we have used the relationship

$$\Gamma(\tfrac{1}{2}-n)\Gamma(\tfrac{1}{2}+n)=\frac{\pi}{\cos(n\pi)}. \tag{29}$$

On the other hand, the coefficient of $e^{i(2n+k)\phi}$ in equation (12) is

$$F=\frac{\Gamma(4n+2k+\alpha+\beta+1)(\sin\theta_1\sin\theta_2)^{2n+k}}{2^{4n+2k}(2n+k)!\Gamma(2n+k+\alpha+\beta+1)} \tag{30}$$

where we have used equations (13) and (23) with $r=p=2n+k$.

Equating equations (28) and (30) we immediately arrive at the summation rule given by equation (14).

Similarly, we compare the coefficients of $e^{im\phi}$ when $m=0$ in equations (6) and (12). In equation (6) this coefficient is

$$F_0=\sum_L A(L,0)(\cos\theta_1\cos\theta_2)^{2n+2k}P_{2n+k-L}^{\alpha_1,L+\beta_1}(\cos\theta_1)P_{2n+k-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2). \tag{31}$$

Using equation (21) we notice that the factorial $(r+m-2n-2k-2j)!$ appearing in its denominator, when $m=0$, dictates the values $k=0$, $r=2n$, $j=0$, and hence $s=0$. This means that it is no longer necessary to distinguish between the odd and even cases, hence the replacement of $2n$ by n. Equation (41) is then written as

$$\begin{aligned}F_0=&\frac{(2n+\alpha_2+\beta_2+1)(-1)^n\Gamma(L+\beta_2+1)\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)\Gamma(n+\beta_2+1)}(\cos\theta_1\cos\theta_2)^n\\&\times\sum_L(n-L)!\sum_x\frac{(-1)^x\Gamma(n+L+\alpha_2+\beta_2+x+1)}{x!(n-L-x)!\Gamma(2L+\alpha_2+\beta_2+x+2)}\\&\times P_{n-L}^{\alpha_1,L+\beta_1}(\cos\theta_1)P_{n-L}^{L+\alpha_2,L+\beta_2}(\cos\theta_2).\end{aligned} \tag{32}$$

The summation over x could be carried out as before to give

$$\frac{1}{(n+L+\alpha_2+\beta_2+1)}\delta_{L,n}.$$

Therefore

$$F_0=\frac{(-1)^n\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}(\cos\theta_1\cos\theta_2)^n. \tag{33}$$

On the other hand, the coefficient of $e^{i0\phi}$ in equation (12) is

$$F_0=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}\sum_{rp}\frac{(-1)^{r+p}r!\Gamma(n+r+\alpha+\beta+1)(2p+1)}{(n-r)!(r-p)!(r+p+1)!\Gamma(r+\alpha+1)}P_p(\cos\theta_1)P_p(\cos\theta_2) \tag{34}$$

where we have used equation (2.5.29) in [4] as

$$Y_n^m(\theta,\phi)=(-1)^m\left(\frac{(2n+1)(n-m)!}{4\pi(n+m)!}\right)^{1/2}P_n^m(\cos\theta)\,e^{im\phi}. \tag{35}$$

Equating formulae (33) and (34) we immediately arrive at the summation rule given by equation (15).

Appendix

To evaluate the double summation over x and s appearing in equation (21) we use equation (A1.1) in [5] as

$$\sum_y \frac{1}{y!\Gamma(m-r+y+1)\Gamma(r-y+1)\Gamma(n-y+1)} = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)\Gamma(r+1)\Gamma(n+m-r+1)}. \tag{A1}$$

Choosing $m=s$, $n=L+\alpha_2+x$ and $r=-L-\beta_2-s-1$ we write equation (A1) as

$$\frac{\Gamma(L+\alpha_2+s+x+1)}{s!\Gamma(L+\alpha_2+x+1)\Gamma(2L+\alpha_2+\beta_2+2s+x+2)} = \sum_y \frac{\Gamma(-L-\beta_2-s)}{y!\Gamma(L+\beta_2+2s+y+1)\Gamma(-L-\beta_2-y-s)} \frac{1}{\Gamma(L+\alpha_2+x-y+1)}.$$

Hence we can write our double summation as

$$\begin{aligned}
&\sum_{sx} \frac{(-1)^{s+x}2^{2s}\Gamma(2n+k+L+\alpha_2+\beta_2+x+1)\Gamma(L+\alpha_2+s+x+1)\Gamma(L+\beta_2+s+1)}{s!(j-s)!x!(2n+k-L-x)!\Gamma(L+\alpha_2+x+1)\Gamma(2L+\alpha_2+\beta_2+2s+x+2)} \\
&\quad = \sum_y \frac{1}{y!} \sum_x \frac{(-1)^x\Gamma(2n+k+L+\alpha_2+\beta_2+x+1)}{x!(2n+k-L-x)!\Gamma(L+\alpha_2+x-y+1)} \\
&\qquad \times \sum_s \frac{(-1)^s 2^{2s}\Gamma(-L-\beta_2-s)\Gamma(L+\beta_2+s+1)}{(j-s)!\Gamma(L+\beta_2+2s+y+2)\Gamma(-L-\beta_2-s-y)} \\
&\quad = \frac{(-1)^{k-L}(\pi)^{1/2}\Gamma(2n+k+L+\alpha_2+\beta_2+1)}{2^{L+\beta_2+1}(2n+k-L)!} \\
&\qquad \times \sum_y \frac{(-1)^y\Gamma(2n+k+\beta_2+y+1)}{2^y y!\Gamma(2n+k+\alpha_2-y+1)\Gamma(L+\beta_2+y+1)} \\
&\qquad \times \sum_s \frac{(-1)^s\Gamma(L+\beta_2+s+y+1)}{(j-s)!\Gamma[\frac{1}{2}(L+\beta_2+y+2)+s]\Gamma[\frac{1}{2}(L+\beta_2+y+3)+s]} \\
&\quad = \frac{\sqrt{\pi}\,(-1)^{k-L+j}\Gamma(2n+k+L+\alpha_2+\beta_2+1)}{(2n+k-L)!(\frac{1}{2}-j)2^{L+\beta_2-2j}} \\
&\qquad \times \sum_y \frac{(-1)^y\Gamma(2n+k+\beta_2+y+1)}{2^y y!\Gamma(2n+k+\alpha_2-y+1)\Gamma(2j+L+\beta_2+y+2)}
\end{aligned} \tag{A2}$$

where we have used equations (A1.1) and (A1.2) in [5], the duplication formula for the gamma function as well as the relationship

$$\frac{\Gamma(n-z)}{\Gamma(-z)} = \frac{(-1)^n z!}{(z-n)!}. \tag{A3}$$

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